Integer value of expression with radicals.
Find all nonnegative real numbers x for which
$\sqrt[3]{13+\sqrt{x}}+\sqrt[3]{13-\sqrt{x}}$
is an integer.
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Let $t := \sqrt[3]{13+\sqrt{x}}+\sqrt[3]{13-\sqrt{x}}$ .Then the problem is:
Find all nonnegative real numbers x for which $t = \sqrt[3]{13+\sqrt{x}}+\sqrt[3]{13-\sqrt{x}} \in \mathbb{Z}$.
We have $t = \sqrt[3]{13+\sqrt{x}}+\sqrt[3]{13-\sqrt{x}} \Leftrightarrow t^3 = 26+3t\sqrt[3]{13+\sqrt{x}}\cdot\sqrt[3]{13-\sqrt{x}} \Leftrightarrow$
$t^3 = 26+3t\sqrt[3]{13^2-x} \Leftrightarrow \left(\frac{t^3-26}{3t}\right)^3 = 13^2-x \Leftrightarrow$
(**1**) $x = 13^2-\left(\frac{t^3-26}{3t}\right)^3$.
where latter equality implies
(**2**) $\frac{t^3-26}{3t} \le \sqrt[3]{13^2}$
Since for any integer t which satisfies to inequality (**2**) formula (**1**) give us value x for which value of expression $\sqrt[3]{13+\sqrt{x}}+\sqrt[3]{13-\sqrt{x}}$ is t then remains to solve inequality (**2**) in integers.
Consider two cases.
1. $t<0$. Then by replacing t with $-t$ we obtain inequality $\frac{t^3+26}{3t} \le \sqrt[3]{13^2}$,where $t \in \mathbb{N}$.
But since by $\frac{t^3+26}{t} = t^2+2\cdot\frac{13}{t} \ge 3\sqrt[3]{t^2\cdot\left(\frac{13}{t}\right)^2} = 3\cdot\sqrt[3]{13^2}$ and equality occurs iff $t = \sqrt[3]{13} \notin \mathbb{N}$ then $\frac{t^3+26}{3t} > \sqrt[3]{13^2}$ for any $t \in \mathbb{N}$.
Thus, there are no integer $t<0$ that satisfies to inequality $\frac{t^3-26}{3t} \le \sqrt[3]{13^2}$.
2. $t>0$. Then $\frac{t^3+26}{3t} \le \sqrt[3]{13^2} \Leftrightarrow t^3-3\cdot\sqrt[3]{13^2}\,t+26 \le 0$. By replacing u in $u^3-3u-2 = (u-2)(u+1)^2$ with $t/\sqrt[3]{13}$ we obtain
$t^3-3\cdot\sqrt[3]{13^2}\,t+26 = \left(t+\sqrt[3]{13}\right)^2\left(t-2\sqrt[3]{13}\right)$
and, therefore, in natural t we have $t^3-3\sqrt[3]{13^2}\,t+26 \le 0 \Leftrightarrow t \le 4$
because $\left\lfloor 2\sqrt[3]{13} \right\rfloor = 4$.
Thus, only $x = 13^2-\left(\frac{t^3-26}{3t}\right)^3$, *where* $t = 1,2,3,4$ *are solutions of the problem*, *namely* $x = 13^2-\left(\frac{1^3-26}{3\cdot 1}\right)^3 = \frac{20188}{27}$, $x = 13^2-\left(\frac{2^3-26}{3\cdot 2}\right)^3 = 196$,
$x = 13^2-\left(\frac{3^3-26}{3\cdot 3}\right)^3 = \frac{123200}{729}$, $x = 13^2-\left(\frac{4^3-26}{3\cdot 4}\right)^3 = \frac{29645}{216}$.